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H_∞ Robust Control via Singular Value Decomposition as a Design Tool for Continuous Dynamic Systems

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Abstract

We present a novel generic tool to design the shape and location of an actuator for continuous elastic dynamic systems, i.e. essential properties of the actuators in order to generate a desired state profile. The main idea of the research is to generate an approximation via reduction of the number of actuators by using the singular value decomposition (SVD). SVD is a powerful and elegant method for data analysis aimed at obtaining low-dimensional approximation of high-dimensional data. We implement our work on the structural dynamics of a clamped elastic beam. By the use of Finite Difference Method (FDM), we divide the beam into discrete elements. Each element has the ability to translate and rotate with respect to the surrounding elements. By implementing the theory of robust H_∞ control, we obtain the optimal control law with respect to the worst exogenous input. This and the use of SVD enables us to approximate efficiently the number of actuators needed. Thus enabling us to reduce the number of actuators that are necessary in order to obtain a desirable state profile with a robust control law.

Keywords: Finite Difference Method, Singular Value Decomposition, H_∞ control

1. INTRODUCTION

Control of continuous elastic dynamic systems is a very important issue in aerospace engineering and structural engineering. The question of minimal number of actuators arises when dealing with large scale systems. Trying to implement a large number of actuators and controlling them in real time is a very difficult problem. We try in our work to reduce the large number of actuators, in order to control a large scale system with multiple inputs.

Various vibration control methods have been studied, which can be categorized into two major groups: passive vibration control and active vibration control. In passive vibration control, passive elements are used to change the system damping and stiffness in order to reduce structural vibration. Although no power source is needed, the dynamics of the plant is often changed, and the weight of the whole system is often increased which is not acceptable in aerospace applications. Furthermore, the structural vibration is only reduced in certain frequencies, with passive vibration control. Due to the limitation of passive vibration control, active vibration control was introduced and there has been a great deal of interest in the active vibration control of structures. The structures of active vibration control, with many actuators and sensors, have been made possible by the use of piezoelectric ceramic and piezopolymer film materials as the sensing and actuating devices [1]. Active vibration control is capable of performing over a broad range of operating conditions, and has the advantage of reduced weight over passive damping methods [1].

One of the earliest works in the field of active vibration and acoustic control was published by Fuller [2]. Feed-forward control was used to reduce narrow band acoustic radiation with structural actuators, and considerable noise attenuations were achieved with this approach [3, 4]. Swigert and Forward used piezoelectric element (PZT) as the active damper to control the mechanical vibration of an end-supported mass [5]. Bailey and Hubbard developed the active vibration control system for a cantilever beam using Poly Vinylidence Flouride (PVDF) [6]. Choi performed vibration control with multi-step Bang-Bang control [7]. Baumann and Eure used feedback control to reduce stochastic disturbances such as turbulent boundary layer noise [8, 9]. Although active control has been used to reduce structural vibrations for many years [2, 10, 11], the application of active vibration control on large-scale systems has achieved little success due to the scalability limitations of traditional centralized control architectures. In general, one controller processes all sensor data to generate optimal actuator inputs in order to reduce the structural vibrations. Thus, there is an overwhelming, some even impractical, computational burden on the centralized controller, when large-scale systems are considered. As a result recent advances in Micro-Electro-Mechanical systems (MEMS) and embedded system technologies have enabled the applications of distributed control designs [12], which is more scalable compared with centralized control and suitable for large-scale systems. A distributed control system normally consists of numerous localized controllers called nodes. Each localized controller has a sensor, an actuator and a means of communicating with other controllers in the system [13, 14]. Therefor there is an urge to reduce the number of actuators and controllers.

We present a method that reduces the number of actuators, such that it approximates the process in the best possible way in the sense of minimizing the Frobenius norm. One of the methods which approximate representation of high-dimensional processes is the singular value decomposition. Singular value decomposition (SVD) is a method of data analysis aimed at obtaining low-dimensional approximation descriptions of high-dimensional processes. In other words in order to obtain a desired damped vibration, the object is divided into n models, elements. Similar to the finite element and difference method [15, 16], i.e. n actuators are required to be controlled in order to obtain the objective damped vibration. By use of SVD and implementation of our method one can achieve an optimal approximation of the objective damped vibration profile by controlling a minimal number of actuators. This way one can analyze large scale models and reduce the number of actuators. The SVD has been used to obtain low-dimensional descriptions of turbulent fluid flows [17], structural vibrations [18, 19], insects gaits [20], and for damage detection [21]. It has also been extensively used in image processing, signal analysis and data compression.

The paper is organized as follows: Section two introduces some notations and reviews regarding SVD and linear H_∞ control. Section three describes the dynamical model. Section four presents the control algorithm for reduction of the number of actuators for large scale systems, implemented on a one side clamped cantilever beam. Section five presents simulation results and section six presents conclusions and future work.

2. REVIEW OF SVD AND H_∞ CONTROL

2.1. Singular Value Decomposition

The reduced order approach by SVD is based on projecting the dynamical system onto subspaces consisting of basis elements that contain characteristics of the expected solution. This is in contrast to the finite element methods, where the elements of the subspaces are uncorrelated to the physical properties of the system that they approximate. We apply the SVD to derive a Galerkin approximation in the spatial variable, with basis vectors corresponding to the solution of the physical system at pre-specified time instances.

Lemma 2.1.1. *: For any matrix $\mathcal{H} \in R^{m\times n}$ exists a unitary matrix, $U^{-1} = U^T$, $U \in R^{m\times m}$ which its columns form an orthonormal basis, a unitary matrix $V \in R^{n\times n}$ which its columns form an orthonormal basis and a diagonal matrix decreasing order $\Sigma \in R^{m\times n}$ i.e.,*

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ \vdots & \sigma_2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \end{bmatrix} \quad ; \quad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m \tag{2.1}$$

such that,

$$\mathcal{H} = U\Sigma V^T \tag{2.2}$$

The diagonal entries σ_i of Σ are called the singular value of $\mathcal{H}$. Σ's elements which are the singular values, are the square roots of the eigenvalues of UU^T and U^TU. The columns of U are called *left singular* *values* of $\mathcal{H}$ which are the eigenvectors of $\mathcal{H}^T\mathcal{H}$. The columns of V are called *right singular values* and are the eigenvectors of $\mathcal{H}\mathcal{H}^T$. Using the orthogonality of V we can write the SVD in the form:

$$\mathcal{H}V = U\Sigma \tag{2.3}$$

We can interpret Eq.(2.3) as a mapping of a special set of orthonormal vectors, columns of V, into an orthonormal set of vectors, columns of U. If we denote $U = [u_1, \ldots, u_n]$, $\Sigma = diag(\sigma_1, \ldots, \sigma_n)$ and $V = [v_1, \ldots, v_n]$, then $\mathcal{H}$ can be written as:

$$\mathcal{H} = U\Sigma V^T = \sum_{i=1}^{n} \sigma_i u_i v_i^T \tag{2.4}$$

This form is called the *dyadic decomposition* of $\mathcal{H}$ which decomposes the matrix $\mathcal{H}$ of rank n into a sum of n matrices of rank one.

2.2. Review of Linear H_∞ State Feedback Control

In order to obtain a control law which is robust to exogenous inputs, we consider the following LTI system described by:

$$\Sigma: \quad \begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + Du \\ y &= C_2 x \quad , \quad C_2 = I \end{aligned} \tag{2.5}$$

where,
x$\in \mathbb{R}^n$ is the state vector,
$w \in \mathbb{R}^q$ is an external disturbance signal,
$u \in \mathbb{R}^m$ is the control input signal,
$z \in \mathbb{R}^r$ is the control output signal,
$y \in \mathbb{R}^p$ is the measurement output signal.

Theorem 2.2.1. *The stability and achievable L_2-gain properties for the linear system Eq. (2.5) can be established by finding a positive definite Lyapunov function $S(x) \triangleq x^T W^{-1} x$ such that the following conditions are satisfied.*

$$\begin{bmatrix} AW + WA^T + B_2 Y + Y^T B_2^T & B_1 & WC_1^T + Y^T D^T \\ * & -\gamma^2 I & 0 \\ * & * & -I \end{bmatrix} \leq 0$$

$$S(x) > 0 \tag{2.6}$$

where $Y = KW$.

Definition 2.2.1. *We say that Eq. (2.5) is L_2-gain stable if there exist $\gamma > 0$ and a locally bounded function $S(x)$, called a storage function, $S(x) \geq 0$, $S(0) = 0$, $\forall x \in \mathbb{R}^n$ and for each admissible input such that $\forall u(\cdot) \in \mathcal{U}$, $\forall t \geq 0$*

$$S(x(t_1)) \leq S(x(t_0)) + \int_{t_0}^{t_1} s(w(t), z(t))dt \tag{2.7}$$

where

$$s = \frac{1}{2}(\gamma^2 ||w||_2^2 - ||z||_2^2) \ \ ,\gamma \geq 0 \tag{2.8}$$

or equivalently,

$$||G||_\infty \triangleq \sup_{||w||_2^2 \neq 0} \frac{||z||_2^2}{||w||_2^2} \leq \gamma \tag{2.9}$$

where G denotes the transfer function of Eq. (2.5) which corresponds to the ratio between the norm of z, control output, to the norm of w, external disturbance [22].

Proof. A state space system Σ is said to be ***dissipative*** with respect to the supply rate function s if there exists a function $S : X \longrightarrow \mathbb{R}^+$, called the storage function, such that for all $x_0 \in X$, at all $t_1 > t_0$, and all input functions w,

$$S(x(t_1)) \leq S(x(t_0)) + \int_{t_0}^{t_1} s(w(t), z(t))dt \tag{2.10}$$

where $x(t_0) = x_0$, and $x(t_1)$ are the state of Σ at time t_0 and t_1 resulting from initial condition x_0 and input function $w(\cdot)$.

Note that whenever the function $S(x(t))$ is differentiable as a function of time, then Eq.(2.10) can be equivalently written as

$$\dot{S}(t) \leq s(t) \tag{2.11}$$

If Σ is dissipative with respect to the supply rate function,

$$s = \frac{1}{2}(\gamma^2 ||w||_2^2 - ||z||_2^2) \ \ ,\gamma \geq 0 \tag{2.12}$$

By using the relations above, dissipative approach, we can conclude that a sufficient condition to the existence of the L_2-gain criterion, is the existence of a function $S(x) \geq 0 \quad \forall \quad x \neq 0$ such that,

$$\dot{S} \leq \frac{1}{2}(\gamma^2 ||w||^2 - ||z||^2) \tag{2.13}$$

by implementing the system to (2.13) we obtain,

$$\begin{aligned} \dot{S} = S_x \dot{x} &\leq \frac{1}{2}(\gamma^2 ||w||^2 - ||z||^2) \\ J \equiv S_x(Ax + B_1 w + B_2 u) &+ \frac{1}{2} z^T z - \frac{1}{2}\gamma^2 w^T w \leq 0 \end{aligned} \tag{2.14}$$

where J is the cost function. The worst input disturbance for maximizing J is given by:

$$w^* = \gamma^{-2} B_1^T S_x^T \tag{2.15}$$

and the optimal controller for minimizing J is,

$$u^* = -(D^T D)^{-1} B_2^T S_x^T$$

where (*) indicates optimality. Substituting this relations into J yields,

$$\begin{aligned} S_x A x + \frac{1}{2}\gamma^{-2} S_x B_1 B_1^T S_x^T \\ - \frac{1}{2} S_x B_2 R^{-1} B_2^T S_x^T + \frac{1}{2} x^T Q x \leq 0 \end{aligned} \tag{2.17}$$

Now, choosing the storage function S to be

$$S = \frac{1}{2} x^T P x \ \ , \ P = P^T \geq 0, \tag{2.18}$$

yields the following *Riccati Inequality* for P:

$$\begin{aligned} &PA + A^T P + Q \\ &+ P\left(\gamma^{-2} B_1 B_1^T - B_2 R^{-1} B_2^T\right) P \leq 0 \end{aligned} \tag{2.19}$$

An equivalent expression is given by:

$$AW + WA^T + \gamma^{-2} B_1 B_1^T - B_2 R^{-1} B_2^T + W C_1^T C_1 W \leq 0 \tag{2.20}$$

where $W \triangleq P^{-1}$. According to the *Schur complement*, relation Eq.(2.20) can be transformed into the following matrix representation,

$$\begin{bmatrix} AW + WA^T - B_2 R^{-1} B_2^T & B_1 & W C_1^T \\ * & -\gamma^2 I & 0 \\ * & * & -I \end{bmatrix} \leq 0 \tag{2.21}$$

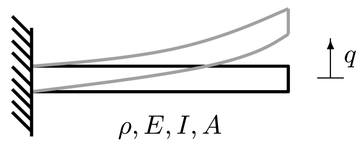


FIG. 1: Illustration of the elastic beam.

The latter inequality is called the *Bounded Real Lemma*. By substituting the state feedback control $u = Kx$ to Σ yields the following closed loop system:

$$\begin{aligned} \dot{x} &= (A + B_2 K)x + B_1 w \\ z &= (C_1 + DK)x \end{aligned} \tag{2.22}$$

and the corresponding LMI (Linear Matrix Inequality),

$$\begin{bmatrix} (A + B_2 K)W + W(A + B_2 K)^T & B_1 & W(C_1 + DK)^T \\ * & -\gamma^2 I & 0 \\ * & * & -I \end{bmatrix} \leq 0 \tag{2.23}$$

which completes the proof $\square$

Note, the following matrix can be solved by the LMI toolbox using MATLAB. The following LMI provides a feasibility test, parameterized in $\gamma > 0$. The L_2 *gain* of the system exists, if and only if exists a K such that the LMI where the upper bound for the external disturbance holds.

3. DYNAMICAL MODEL

In this section we present the dynamical analysis of an elastic beam Fig. (1). By using the following PDE (Partial Differential Equation) according to Euler-Bernoulli beam theory assumptions, one can represent the elastic equation of the beam [23],

$$\rho A \frac{\partial^2 w(q,t)}{\partial t^2} + EI \frac{\partial^4 w(q,t)}{\partial q^4} = \mathcal{F}(q,t), \tag{3.1}$$

where $w(q,t)$ denotes the beam's vertical displacement, ρ is the beam's density, A is the beam's cross section, E is the Young's modulus of elasticity, I is the moment of inertia and $\mathcal{F}$ denotes a distributed force acting along the beam which denotes the inputs u. The boundary conditions, accounted for a cantilever beam,

$$w(q,t) = \frac{\partial w(q,t)}{\partial q} = 0 \quad ,\text{at } q=0, \qquad (3.2)$$
$$\frac{\partial^2 w(q,t)}{\partial q^2} = \frac{\partial^3 w(q,t)}{\partial q^3} = 0 \quad ,\text{at } q=L,$$

where L is the beam's length. Since there is no convenient solution to the system, an efficient explicit second-order accurate finite differences scheme is proposed:

$$\frac{w_s^{m+1} - 2w_s^m + w_s^{m-1}}{\Delta t^2} = \qquad (3.3)$$
$$-b^2 \frac{w_{s+2}^m - 4w_{s+1}^m + 6w_s^m - 4w_{s-1}^m + w_{s-2}^m}{\Delta q2} + c\mathcal{F}_s^m$$

where $b^2 = (EI)/(\rho A)$, $c = 1/(\rho A)$ and m, s represents discrete space and time respectively. In order to maintain stability of the scheme, it requires that [24],

$$b\mu \leq \frac{1}{2} \qquad (3.4)$$

where $\mu = \Delta t/\Delta q^2$. Eq.(3.3) can be written in a matrix form given by

$$w(m+1) = Aw(m) + Bu(m) - w(m-1), \qquad (3.5)$$

where $B = diag(\Delta t^2 c)$, and A is a diagonal matrix whose elements are,

$$A = diag(-b^2\mu^2, 4b^2\mu^2, -6b^2\mu^2 + 2, 4b^2\mu^2, -b^2\mu^2) \qquad (3.6)$$

By rearranging the finite differences equation in the form of a discrete state space representation we obtain:

$$\begin{bmatrix} q_1(m+1) \\ q_2(m+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & A \end{bmatrix} \begin{bmatrix} q_1(m) \\ q_2(m) \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} u(m), \qquad (3.7)$$

where $q_1(m) = w(m-1)$, $q_2(m) = w(m)$.

4. CONTROL ALGORITHM

We implement the SVD algorithm by applying it on the controllers outputs. The columns of V can be interpreted as inputs, the diagonal values of the matrix σ can be interpreted as the controls gain and the columns of U can be interpreted as outputs.

$$u^{n\times t} \approx \tilde{u}^{n\times t} = U^{n\times k}\sigma^{k\times k}V^{k\times t} \qquad (4.1)$$

As a result we are interested only in the outputs, the columns of U. We use only k columns that correspond to the largest singular values, i.e. the first k^{th} columns. These columns represent the optimal eigenvectors, in the sense of minimal number and maximal influence. The k columns of U correspond to the most influenceable system inputs. These are actually the main directions of the acting forces, which act on the body. It can be interpreted as the shapes of the actuators.

After the robust control law is obtained by the use of the linear H_∞ controller, we minimize the number of actuators by using the SVD algorithm. In order to verify the algorithm we use the first k^{th} columns of U to control the system. We obtain the approximated system by implementing again the linear H_∞ controller on the reduced order inputs,

$$X_{apx}(m+1) = \tilde{A}X_{apx}(m) + \tilde{B}_{apx}u_{apx}(m) \qquad (4.2)$$

where,

$$\tilde{A} = \begin{bmatrix} 0 & 1 \\ -1 & A \end{bmatrix}, \quad \tilde{B}_{apx} = \begin{bmatrix} 0 \\ B \end{bmatrix} U^{n\times k} \qquad (4.3)$$

i.e. u_{apx} is an approximation based on the transformation of $\sigma^{k\times k}V^{k\times t}$ by the operator u. By applying the linear H_∞ controller on $\tilde{A}$ and $\tilde{B}_{apx}$ we obtain u_{apx}.

In summary, an algorithm to solve the linear H_∞ problem for large scale system with a minimal number of actuators is:

1. Conversion of the continuous state space dynamical system to a discrete one.
2. Determining the control input as a function of time and space for the large scale system.
3. Computing the SVD matrices.
4. Choosing the first k most influenceable columns of U, i.e. obtaining the new actuators.
5. Recalculating the controller for the approximated actuated system.
6. Implementing the new controller on the system where only k actuators are used.

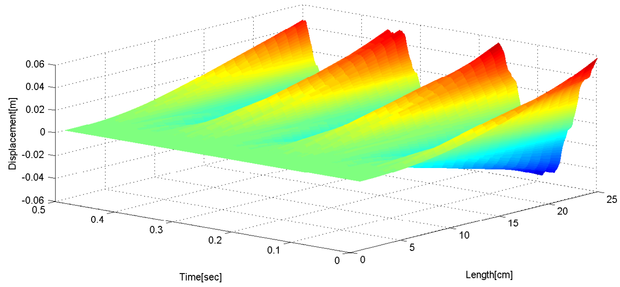


FIG. 2: The 25 cm beam, oscillating around zero, for a period of 0.5 seconds, while the controller is turned off.

We choose the order of approximation according to the largest singular values. We choose the k largest singular values according to the value of the norm corresponding to an acceptable nominal value.

5. SIMULATIONS

In order to simulate the algorithm we chose a metal beam with a density of $\rho = 7850\ kg/m^3$ and a Young's modulus of $E = 200\ GPa$. The length of the beam is 24 cm and has a cross section of 0.125 cm^2 with a moment of inertia $I = 1.76 \cdot 10^{-5}\ cm^4$. We chose an initial condition of $w(q, 0) = q^2\ cm$ and divided the beam into 25 equal elements. We considered the external disturbance as a *bounded* standard Gaussian function, where the disturbance matrix is unitary. We start by demonstrating the dynamic behavior of the beam while the controller is turned off and the plant has no damping Fig. (2). Fig (3) presents the displacement of the beam while using all 25 actuators in order to control the beam where $\gamma = 10.1$. Fig (4) presents the displacement of the beam while using only three actuators in order to control the beam where $\gamma = 13.3$. It can be seen that the closed loop dynamic behavior is very similar to the dynamic behavior where all twenty five actuators were applied. Despite the fact that there is a slight undershoot, while there was non when all 25 actuators are used. Fig (5) presents the weights of the actuators which are applied for a third degree approximation. We can consider the bars as the shapes of the actuators which are needed in order to obtain an approximated solution. The control inputs are represented in Fig. (6) respectively to the actuators which are presented in Fig. (5).

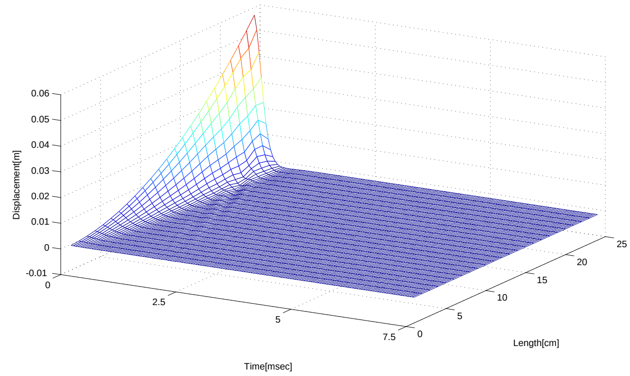


FIG. 3: The 25 cm beam, regulated while the controller is turned on and acts on all the 25 actuators.

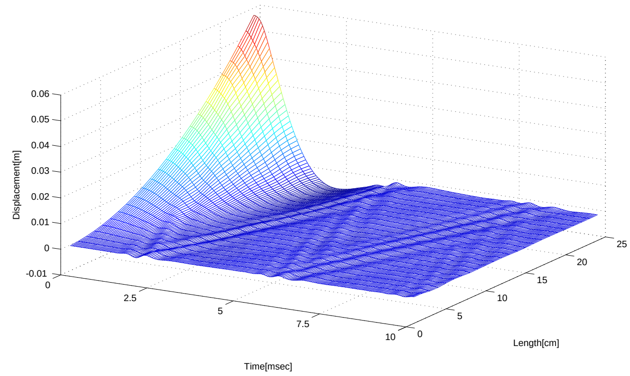
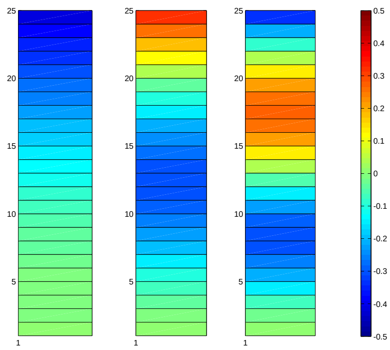


FIG. 4: The 25 cm beam, regulated while the controller is turned on and only three actuators are controlled.

6. CONCLUSIONS

We presented in this paper a novel algorithm for reducing the number of actuators for large scale systems in closed loop. The algorithm was based on the use of the singular value decomposition and the



(a) (b) (c)

FIG. 5: Representation of the discrete weights of the actuators, i.e. shapes, which are applied on the third degree approximated system.

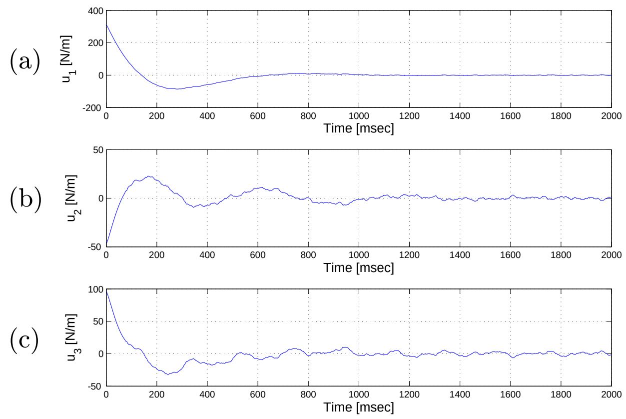


FIG. 6: Control inputs under an initial condition. Each control input is correlated to each of the actuators a,b,c respectively.

robust H_∞ controller. We have shown that it is possible to control a large scale system by significantly reducing the number of actuators. We implemented our work on a two dimensional steal beam. In the future we plane on expanding our work to nonlinear problems while solving the output feedback robust H_∞ controller and implementing the algorithm on three dimensional problems.

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